Student Preferences in the Design of Worked Solutions in Undergraduate Mathematics

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The ubiquitous practice of providing worked solutions to exercises in mathematics education has been under-researched. Little is known about what elements of a worked solution are valued by students. This exploratory study sought in-depth feedback from six undergraduate students who experienced a range of worked solutions designed to encourage engagement. Elements that were valued included detail, explanation, and choice. Elements of worked solutions with which students did not engage were extension and reflection tasks.

Pedagogical practices define teaching and, to a large extent, the learning experience. They capture not only what teachers do but also embody the judgements and decisions they make concerning students and their learning needs (Foley, 2007, as cited in Churchill et al., 2011). Worked solutions to home study exercises are one pedagogical practice that mathematics teachers in both schools and universities commonly use. They are expert solutions that differ in detail, prompts and explanation, in presentation, and in the engagement required from students. Despite their pervasiveness and diversity, little research has been undertaken in the use of worked solutions. Only one study (Aminifar, Porter, Caladine, & Nelson, 2007) could be found that has researched student use and perceptions of worked solutions.

The qualitative research presented here is part of a larger exploratory study at an Australian university that examined, from the student perspective, the pedagogical practice of worked solutions in first year undergraduate mathematics (Mendiolea, 2012). The findings reported here are in response to the research question: What elements of worked solutions do undergraduate mathematics students value?

Unlike the study by Aminifar et al. (2007), which sought student perceptions about one solution format, a step-by-step video recording of worked solutions, this study had a wider scope. To serve as a stimulus for identifying preferences in the way worked solutions are provided, students experienced a set of six different solution formats for the exercises concerning one topic (partial fractions) in their first semester curriculum. The design took into account the common errors of the previous year's cohort and drew on the literature concerning constructivist theories of learning and literature investigating worked examples in mathematics which, unlike worked solutions, has attracted substantial research interest.

Literature Review

Learning theories applicable to the teaching and learning of mathematics

Constructivist theories of learning are a powerful driving force in mathematics education. Goos, Stillman, and Vale (2007) stress that constructivism gives priority to the individual construction of mathematical understanding. This implies that the design of worked solutions needs to recognise the diversity of the student needs and needs to encourage students to construct their own understanding.

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Perkins (1991) notes that constructivism demands that learners take agency of their own learning. Pedagogical approaches that foster agency offer students opportunities to make choices that are powerful enough to engage them (Stefanou, Perencevich, DiCintio, & Turner, 2004). The question that then needs to be considered is whether students can make these choices appropriately and in recognition of their own needs. Grow (1991) urges caution in this regard, suggesting that a complex collection of metacognitive and learning skills are needed which not all learners may have. Renkl (1999, p. 477) argues that a lack of successful cognitive engagement in learning may lead to "illusions of understanding". Inappropriate use of worked examples (or the closely related worked solutions) may result in this "illusion of understanding" if they require learners to do little, cognitively speaking, when solving problems (Renkl, 1999).

The use of extension activities is one way of accommodating high achievers. These activities facilitate the greatest amount of enjoyment for high achievers where, conversely, low achievers tend to avoid this challenge (Nakamura, 1988, as cited in Middleton & Spanias, 1999). Many students are only concerned with the accuracy of their work and what they need to do to receive the grade they desire (Middleton & Spanias, 1999) and thus do not value extension tasks. Therefore, only some students engage with extension tasks.

Learning from worked examples

A consistent finding in studies that have examined the effectiveness of worked examples in classroom instruction (for example, Reisslein, Sullivan, & Reisslein, 2007; Renkl, Atkinson, Maier, & Staley, 2002) is that there are elements of worked examples which can be manipulated to encourage learning. In a review of the literature on worked examples, Shen and Tsai (2009) summarised eight principles for the design of effective and supportive worked examples, all of which were devised with the aim of engaging students in learning. Four of these principles – self-explanation, completion, fading and process – were of particular relevance to the design of the worked solutions used in this investigation.

The self-explanation principle was first identified in the research of Chi et al. (1989) when the authors showed that the extent to which learners profited from the study of worked examples depended on the quality of the independent rationalisations and generalisations they made about the example they were studying. Further investigation appeared to demonstrate that self-explanation fosters the integration of new knowledge with existing knowledge and, thus, supports learners as they update and refine their models of understanding (Chi, de Leeuw, Chiu, & LaVancher, 1994). Chi, Bassok, Lewis, Reimann, and Glaser (1989) found that the more successful learners also had a greater metacognitive awareness of their problems and addressed them when using worked examples.

Reflection is closely linked to the self-explanation principle in worked example research. It is an active and self-directed process, consisting of exploration and discovery, which can be focussed into productive learning experiences for students (Boud, Keogh, & Walker, 1985). Schön (1983) identified two distinct phases of reflection in the learning process: reflection-*in*-action and reflection-*on*-action. Further work by Boud and Walker (1991) implied a third stage of reflection, reflection-*before*-action, although the authors did not specifically use the term.

The first evidence of the completion and fading principles was gathered by Paas (1992) when he demonstrated that both complete and incomplete worked examples benefit student understanding. Paas (1992) believed that incomplete worked examples allow the scope for more high quality self-explanations to occur.

Renkl et al. (2002) further investigated the concept and demonstrated that it is beneficial to remove steps from the sequence of a worked example. They termed the removal of a section of a worked example as "fading" and this became the basis of the fading principle. Reisslein et al. (2007) varied the pace at which worked example steps were "faded". The key finding of their research was that there is a significant link between the optimal pace of fading and the prior knowledge of a student. By providing students with agency in their learning, they can control the pace of fading.

The process principle works on the assumption that providing learners with sub-goals enhances their ability to solve problems. Research into the process principle arose from the observation by Catrambone and Holyoak (1990) that the use of annotations in examples to highlight sub-goals seemed to increase the likelihood that a learner would modify a known solution method when solving a problem as opposed to applying it without adaptation. Catrambone (1994, 1996) went on to investigate the concept further, consistently finding that sub-goals can be conveyed to learners through worked examples, and that learning these sub-goals helps students achieve success in novel problems.

While the structural similarities between worked solutions and worked examples suggest that the same elements may be useful in designing worked solutions, there are differences between the two pedagogical practices. Firstly, their places in the learning sequence are different. Worked examples are used in the concept demonstration phase, providing a guide of what to expect. In contrast, students encounter worked solutions later in the learning sequence when they, themselves, are attempting the exercises. Secondly, worked solutions are generally used by students independently, without the guidance of a lecturer or tutor. Conversely, worked examples are usually used during face-to-face teaching in lectures or tutorials.

Methodology

Researchers undertake exploratory research work in relatively unstudied areas to develop a general understanding of the area and to inform future research (Adler & Clark, 2008). Exploratory research is nearly always inductive in nature because of the lack of relevant research and guiding frameworks in the areas it investigates (Adler & Clark, 2008). To guide this exploratory research, the research question "What elements of worked solutions do undergraduate mathematics students value?" was developed.

Solution sets each comprising six solution formats, were designed for each of nine partial fractions exercises. The names of the six formats indicate the degree of detail provided: Format 1: *General guidance* (an overview of the solution process), Format 2: *Some guidance* (an overview plus prompting), Format 3: *Detailed guidance* (direct, written instruction), Format 4: *Complete worked out solution* (complete mathematical working), Format 5: *Screencast* (a video playback accompanied by narration) and Format 6: *Extension* (questions intended to extend student knowledge). The first five formats built upon each other while the sixth focused on extending the knowledge of interested students. To encourage the use of reflection before commencing the exercises and upon their completion, all written formats included the same set of reflection questions. All formats articulated the same set of steps involved in arriving at the solution. By providing formats with different levels of support, students were invited to take agency of their learning and make choices based on their needs.

The worked solutions for the other topics in the subject remained unchanged. Their format presented the key steps of the solution with minimal written descriptions or explanations, showing for the most part, only the mathematical working. The online environment that students used for the subject was used in this study to provide the set of solution formats for the partial fraction questions. Students could choose to view one or more formats by clicking the appropriate link.

The participants in the study were interviewed in the first half of the second semester. The mathematics subject lecturer invited continuing students who had indicated that they had used the worked solutions for the partial fractions topic in the first semester to take part in the study—six students accepted the invitation. The characteristics of the participants are outlined in Table 1 below. While there is diversity in terms of age, achievement, degree of study, and gender amongst the six interviewees, a limitation of the study was that no interview data were collected from the students who had withdrawn or failed the first semester subject.

Three semi-structured interviews each approximately one hour long were conducted. The first interview involved two students; the second was with one student; and the third involved three students. Student perceptions were sought regarding the typical worked solutions experienced in university mathematics and the stimulus set of worked solution formats for the partial fractions topic. To encourage reflection on their experience of the changed worked solutions, students were provided copies of the six solution formats for two of the nine exercises.

Table 1
Interview participants

Pseudonym	Gender	Age	Degree	Result for first semester subject
Erin	F	18	Engineering	Distinction
Nathan	M	28	Engineering	Pass
Tom	M	19	Education	Pass
Casey	F	18	Science (Advanced)	Distinction
Cohan	M	28	Engineering	Distinction
Katie	F	19	Education	Pass

The interview transcripts were emailed to the interviewees for review. No changes were requested. As the researcher transcribed the interviews, a comprehensive view of the data as a whole was gained and this view was kept in mind when analysis began. The data were coded inductively to identify any emerging themes (Braun & Clark, 2006). The consistency of judgements is a distinguishing factor in high-level qualitative research (Lichtman, 2010). To minimise issues of reliability in interpretation, the researcher was assisted by a more experienced researcher. The small number of interviews meant that it was possible to complete this coding manually using an excel spreadsheet for recording purposes.

Results

The participants in the study held similar views regarding what they valued in solutions and what they did not find useful. The participants valued the design features of detail, explanation, process, choice, and succinctness.

Detail: "I think we do need, we definitely need, fully worked solutions for at least some of the questions." – Nathan

All interviewees, regardless of achievement level, valued solutions which are detailed and show every step of mathematical working. Students found it frustrating and time consuming to address the "gaps" they found in solutions when a mathematical step was not shown or shown only in part. At the time of the interview, one student had stopped using worked solutions for that reason. The students interviewed did not expect detailed solutions for every exercise. They recommended that the initial solutions in an exercise set be presented in full and that this support be reduced as the exercise set progressed.

Explanation: "I find a lot of the solutions are, like, obviously you do it like this, and it's not obvious… Explanation makes all the difference." – Cohan

Across the interviews, students expressed a desire to know why particular steps are taken in solving problems. Many participants recommended that annotations and explanations accompany mathematical working in written solutions. Katie, for example, found that she could not follow solutions, and wanted annotations to help her understand the solution. She explained, "I wouldn't mind a little more explanation of the logic. Like just even little sentences: 'Using this proof is how we've gotten to this line', 'This is cancelled because this happens'."

Process delineated: "...without the process, I get kind of lost." – Nathan

While the students tended to consider, remember, and use a procedure in different ways, they all thought that the presence of an obvious procedure containing a set of explicit steps provided peace of mind when solving a question. They explained that it gave them a means to organise their thoughts, a framework to apply to different situations, and something to fall back on when flustered or nervous. Students also tended to recognise the necessity to develop independence from worked solutions, and developing a procedure was often a major part of detaching themselves from worked solutions.

Choice: "I like how it was split up in to the six different types because at school we were just given the answer or worked solution rather than the guidance and that sort of thing." – Casey

Overall, the participants felt that having a choice of worked solution formats, each with a different level of support, was beneficial to their learning. Casey explains:

> If you're struggling with it, I think all six are definitely helpful, especially the first five. Even the sixth step helps to solidify your understanding of it rather than just the process of it ... I think all six are beneficial depending on your ability.

The participants recommended integrating the first four formats together into a single electronic resource within which steps in a process can be shown or hidden based on the needs of the learner. Cohan, who is aware of his inclination to not work through a solution if the final answer is present, also suggested having an additional solution format that has the answer hidden. He states, "If I'm doing the problem, I won't want to look at the answer because [if I do] I'm not going to be using my process".

Succinctness: "I think anything with too much writing I would tend to skip over and even if there is important information in the writing, I would have skipped over it anyway." – Erin

Interviewees stressed the need to moderate the amount of text in worked solutions. While students repeatedly stated a desire for detail, many found the amount of text to be discouraging. This theme was evident in the interviews with Erin, Nathan and Tom.

Tom believed that large amounts of writing would discourage student use, particularly in the formats where no mathematical working was shown. Erin recommended both a reduction in writing in the various solution formats, and an introduction of mathematical working earlier in the sequence of formats. While she thought that the amount of writing was off-putting, she still wanted a written explanation:

> I don't like reading. And I think that was the case for a lot of us in my study group ... Because I think the general tendency for engineers is to avoid reading at all costs, so I think they went straight for the maths formats. But I think they like an explanation, like a sentence or something.

Erin echoed this statement and expressed a desire for graphical annotations to supplement any written ones:

> I think it's really hard to describe maths in English. So it takes me a long time to understand written descriptions of maths. So if it had check powers, and then it had an arrow 'Is this greater than that?' I think I would have responded better to that.

However, while the participants identified several design elements that they felt facilitated their learning, they did not use the reflection prompts nor did they engage with the extension tasks. Further, while the interviewees did express an appreciation for the learning potential of the extension questions, many cited a lack of available time as the major impediment to engaging with them.

Discussion, Implications, and Conclusion

This study indicated that students prefer to have an explicit process to follow when completing mathematics problems. Five of the six interview participants explained that they used the explicit process delineated in the solutions a means to organise their thoughts, a framework to apply to different situations, and something to fall back on when stressed. The internalisation of a process so that it became second nature, or the memorisation of a general procedure to solve a problem, were techniques used to move from solving problems with support from a worked solution to solving problems independently.

The presence of a clear process in all formats in the set was justified by the process principle (Catrambone & Holyoak, 1990) identified in worked example research. This study suggests that the process principle is transferrable to worked solutions.

Many participants in this study also valued detailed solutions which show all steps and offer an explanation of the underlying principles. However, this preference had a qualification: students do not like to read an extended explanation to locate the information they require. In the interviews, participants reported frustration when they encountered "gaps" in solutions where a step was left out. They explained that these "gaps" hindered their learning, an observation that contradicts the completion (Paas, 1992) and fading (Renkl et al., 2002) principles in worked example design.

Students also expressed a desire to know why a step was performed to assist in building their understanding. This reflects the individual construction of mathematical understanding (Goos et al., 2007). One interview participant, however, believed that providing large amounts of written explanation to show why a step was performed could discourage students from using solutions.

Some students also tended to make annotations to articulate their understanding, adding to the detail of the solution. In this way, they demonstrated the ideas of the self-explanation principle (Chi et al., 1989) in worked example research by making their own annotations and continually refining their understanding by self-explaining.

The lack of engagement with the extension exercises is consistent with other research showing that only small numbers of students use extension tasks and most who do are high achievers. None of the participants had achieved the highest level (high distinction) in their first semester mathematics. High achievers tend to attempt extension tasks because the challenge adds to their satisfaction and because they want to understand "why" as well as "how" (Middleton & Spanias, 1999).

The participants had also not engaged with the reflection exercises. Boud et al. (1985) explain that only learners themselves can effectively reflect. Reflective practices can be prompted but, ultimately, any effect they have is limited by the way in which students engage with them. It may have been that the questions intended to prompt self-explanation and reflection in the set of worked solutions were too general, and that more focused guidance would have been more valuable to students.

This study suggests that the challenge for higher education lecturers is to incorporate student preferences when designing worked solutions. Lack of detail and explanation in the solutions may lead to frustration when students are unable to independently and efficiently fill in the "gaps". However, teachers also need to moderate student desire for complete worked solutions. If worked solutions always detail every step, there is no requirement for learners to construct meaning for themselves and no opportunity for students to work independently, which could lead to "illusions of understanding" (Renkl, 1999, p. 478). In addition, the findings suggest that students should be given the opportunity to make choices and take agency of their learning by offering a range of solution options.

The exploratory study reported here was small and limited to investigating the elements of solutions that encourage student engagement. Possible directions for future research include investigating any correlation between performance and the preferred elements identified by students; analysing the manner in which students select solution formats; and exploring further the use of reflection in solution formats.

This investigation was predicated on the assumption that pedagogical practices based on constructivist theories of learning and principles of worked example design could be usefully applied to the design of worked solutions to exercises in mathematics. The study showed that students did value some aspects of the worked solutions but not others. More work is required to develop worked solutions in mathematics that are of value to students.

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